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SELF-SIMILARITY: DIMENSIONAL ANALYSIS, AND INTERMEDIATE ASYMPTOTICS *

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Distinct questions in the theory of self-similar solutions are considered and clarified in this paper (**).

The fundamental criticisms in the review are: 1) separation of self-similar solutions into solutions of the first and second kinds, which is considered the "fundamental idea underlying the book" in the review, and 2) questions of priority: "realizations of the tendency to introduce invented myths about the achievements of some authors with the disparagement of the value and meaning of the results of other authors".

It is emphasized (p.261) that in fact questions of priority govern a "fundamental thesis of the proposed criticism". In addition, several specific comments are made. Let us examine all these statements.

1. Self-similar solutions of the first and second kind. Examples.

For definiteness, let a system of partial differential equations have a unique solution u under certain supplementary (initial, boundary, etc.) conditions. It can be represented in the dimensionless form

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_m) \tag{1.1}$$

$$\Pi = \frac{u}{a_1^{p} \ldots a_k^{r}}, \quad \Pi_1 = \frac{b_1}{a_1^{p_1} \ldots a_k^{r_1}}, \ldots, \Pi_m = \frac{b_m}{a_1^{p_m} \ldots a_k^{r_m}}$$

Here $a_1, \ldots, a_k, b_1, \ldots, b_m$ are independent variables and constant parameters in the equations and the supplementary conditions. We shall consider the dimension of the quantities $a_1 \ldots a_k$ as independent, while the dimension of the quantities $u, b_1, \ldots, b_m$ are expressed by power-law combinations of the dimensions of $a_1, \ldots, a_k$.

Self-similar solutions correspond to zero of infinite values of one or more constant parameters of the problem, which have the dimensions of the independent variables (a point explosion in an infinite medium, an instantaneous point heat source in an infinite rod, a concentrated force acting on the boundary of an elastic half-plane, etc.). Hence in passing to the limit from the non-self-similar to the self-similar solution of a given fixed problem, at least one of the dimensionless parameters, Π_1 for definiteness, will tend to zero or infinity.

Two possibilities exist as Π_1 tends to zero or infinity: either the function Φ tends to a finite, nonzero limit or not. In the former case, the function $\Phi(\Pi_1, \Pi_2, \ldots, \Pi_m)$ can, for sufficiently large (or small) Π_1, be replaced in (1) to any degree of accuracy by its limit value $\Phi({}^{0}_{\infty}, \Pi_2, \ldots, \Pi_m) = \Phi_1(\Pi_2, \ldots, \Pi_m)$, from which we obtain

$$\Pi = \Phi_1(\Pi_2, \ldots, \Pi_m) \text{ or } u = a_1^p \ldots a_k^r \Phi_1(\Pi_2, \ldots, \Pi_m) \tag{2}$$

Thus, the number of arguments is diminished here by one, as compared with the general case (1), whereupon self-similarity of the solution is achieved.

In the latter case, it is generally impossible to do this: if no finite limit, not equal to zero, exists for the function Φ in (1), then the quantity Π_1 remains essential no matter how large or small it may be, and the number of arguments of the function Φ generally cannot be decreased. There is, however, an important exception here. In the simplest case (there is a complete classification in /1/), for small (large) Π_1, let the function Φ behave to infinitesimal accuracy as

$$\Phi = \Pi_1^{\alpha} \Phi_1(\Pi_2, \ldots, \Pi_m) \tag{3}$$

where α is a number governed by the structure of the solution of this problem and generally dependent on the parameters $\Pi_2, \ldots, \Pi_m$, or a part of them. By inserting the expression (3) for Φ into (1) and passing to the limit as $\Pi_1 \to 0, \infty$, we obtain the trivial relationship $\Pi = 0$ or $\Pi = \infty$, i.e., $u = 0$ or $u = \infty$, from which it is impossible to extract a meaningful result. If desired, it is certainly possible to satisfy this relationship. However, it is possible to go futher, and by taking Π_1 sufficiently small (large) but finite, to introduce the

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**) Apropos of the paper by V. V. Markov, "Illegitimate tendencies in the use of the concept of self-similar phenomena", which is a review of the book /1/. This paper is henceforth called the "review". It appears in this Issue, see pp.260-266.

asymptotics (3) into (1) without passing to the limit. Using the notation

$$\frac{\Pi}{\Pi_1{}^{\alpha}} = \Pi_* = \frac{u}{a_1^{p-\alpha p_1} \ldots a_k^{r-\alpha r_1} b_1{}^{\alpha}} \tag{4}$$

we obtain to arbitrary accuracy

$$\Pi_* = \Phi_1(\Pi_2, \ldots, \Pi_m) \text{ or } u = a_1^{p-\alpha p_1} \ldots a_k^{r-\alpha r_1} b_1^{\alpha} \Phi_1(\Pi_2, \ldots, \Pi_m) \tag{5}$$

The relationship (5) is of the same form as (2), and also assures the self-similarity of the solution since the number of arguments of the function Φ has been diminshed by one.

The essential distinction between these two cases is that in the former the structure of the whole solution is found by a simple dimensional analysis, and the parameter b_1 generally vanishes from consideration. In the latter case, the structure of the parameter Π_*, and therefore, of the whole solution cannot be determined by dimensional analysis since the number α is unknown and an additional investigation is required for its determination. Moreover, the parameter b_1 remains essential. However, self-similarity holds in both cases. In order to distinguish them, we call the self-similar solutions corresponding to the former case solutions of the first kind, and to the latter case, of the second kind.

Thus, if self-similar solutions with power-law self-similar variables exist for a given formulation of the problem as a whole (initial, boundary, mixed, etc.), they are obtained from the non-self-similar solutions by passage to the limit as some parameter (parameters) making the solution non-self-similar, tends to zero or infinity. If this passage to the limit yields a finite limit different from zero, then the self-similar solution is called a solution of the first kind(*). If a finite, nonzero limit does not exist, but with the mentioned parameter (parameters) tending to zero (infinity) there is a power-law asymptotics which indeed assures self-similarity of the limit solution, then the self-similar solution is called a solution of the second kind. I cannot conceive how these constructive definitions can be negated.

The above is illustrated by the example mentioned in the review, which is considered in detail in my book and refers to the solution of the Cauchy problem with the initial data

$$u(x, 0) = \frac{Q}{l} u_0\left(\frac{x}{l}\right), \quad \int_{-\infty}^{\infty} u_0(\zeta)\, d\zeta = 1 \tag{6}$$

for the nonlinear (**) equation of heat conduction that is also encountered in the theory of filtration

$$\partial_t u = \begin{cases} \varkappa \partial^2_{xx} u & (\partial_t u \geqslant 0) \\ \varkappa_1 \partial^2_{xx} u & (\partial_t u < 0) \end{cases} \tag{7}$$

Here $u_0(\zeta)$ is a "deltalike" smooth even function which decreases rapidly as $|\zeta|$ grows, $Q, l, \varkappa, \varkappa_1$ are positive constants, and x and t are the space variable and time, respectively. The solutions u evidently depends on the quantities $t, \varkappa, Q, x, l, \varkappa_1$. The dimensions of the first three are independent, and by virtue of dimensional analysis the solution is represented in the form

$$u = \frac{Q}{\sqrt{\varkappa t}} \Phi(\Pi_1, \Pi_2, \Pi_3), \quad \Pi_1 = \frac{l}{\sqrt{\varkappa t}}, \quad \Pi_2 = \frac{x}{\sqrt{\varkappa t}}, \quad \Pi_3 = \frac{\varkappa_1}{\varkappa} \tag{8}$$

We shall now shrink the dimension of the domain of initial heat liberation to zero, $l \to 0$, while leaving all the independent variables and the remaining parameters of the problem invariant. Here $\Pi_1 \to 0$. It turns out that the situation is essentially distinct for the cases $\varkappa_1 = \varkappa$, i.e.., $\Pi_3 = 1$ (the classical linear equation of heat conduction), and $\varkappa_1 \neq \varkappa$, i.e., $\Pi_3 \neq 1$. In the former case, a finite, nonzero limit of the function $\Phi(\Pi_1, \Pi_2, 1)$, equal to $\exp(-\Pi_2{}^2/4)/2\sqrt{\pi}$, exists as $\Pi_1 \to 0$. This limit corresponds to a solution of instantaneous source type for the linear heat-conduction equation

$$u = \frac{Q}{2\sqrt{\pi \varkappa t}} \exp\left(-\frac{x^2}{4\varkappa t}\right) \tag{9}$$

In the latter case, no finite, nonzero limit exists for the function $\Phi(\Pi_1, \Pi_2, \Pi_3)$ as $\Pi_1 \to 0$: the limit is zero or infinity depending on whether the ratio $\Pi_3 = \varkappa_1/\varkappa$ is greater or less than one. It is possible, understandably, to limit oneself to this trivial result.

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*) Nowhere in the book /1/ are solutions of the first kind called "naive solutions" (c.f. p.261 of the review article).

**) Despite what is said in p. 265 of the review article, equation (7) is essentially nonlinear, and therefore, the whole formulation of the problem is nonlinear.

However, one can proceed further. Let us recall that heat liberation at a point is an idealization. In real problems, and particularly in any machine computation, l is finite. Let us pose the question of how the solution will behave for small but finite $\Pi_1 = l/\sqrt{\varkappa t}$? The answer (see /1/) turns out to be meaningful: the function $\Phi(\Pi_1, \Pi_2, \Pi_3)$ has the power-law asymptotics $\Phi \sim \Pi_1^{\alpha}\Phi_1(\Pi_2, \Pi_3)$ for small Π_1 and other arguments fixed, where $\alpha \neq 0$ for $\Pi_3 \neq 1$ $(\varkappa_1 \neq \varkappa)$, and the solution itself has the form

$$u = \frac{A}{(\varkappa t)^{(1+\alpha)/2}} \Phi_1\left(\frac{x}{\sqrt{\varkappa t}}, \frac{\varkappa_1}{\varkappa}\right), \quad A = Ql^{\alpha} \tag{10}$$

As is seen, this solution is also self-similar. However, the self-similarity is not such as in the case of the classical solution (9): the exponent α is not found from dimensional considerations. It is determined by the specific properties of this problem: the analytic behavior of the function Φ for Π_1 near zero. To determine α by starting from the formulation of the Cauchy problem as a whole, **a nonlinear eigenvalue problem** is posed and solved.

Let us note that $\Pi_1 = l/\sqrt{\varkappa t}$ can tend to zero by letting the time t tend to infinity but keeping l invariant. The variable $\Pi_2 = x/\sqrt{\varkappa t}$ can hence also remain invariant by varying x in a suitable manner. Thus it is clarified that (10) is not only the limiting form of the solution as $l \to 0$ but also the asymptotics of a non-self-similar Cauchy problem (6), (7) for finite l and $t \to \infty$.

The question can certainly be posed formally, as is proposed in the review (p.261): let it be required to find the solution of equation (7) that is determined by the parameters in the equation and a constant A of given dimensions $[A] = [u]L^{\beta+1}$, where β is a previously assigned number, and $[u]$ is the dimensions of the solution u. Dimensional analysis actually yields that this solution is represented in the form (10) upon replacement of α by β. However, an attempt to determine the function Φ_1 locally turns out to be inconsistent since by substituting (10) into (7) we do not know the sign of the product $\partial_t u$ for given x and t, and hence, we cannot choose between the coefficients $\varkappa$ and $\varkappa_1$ in (7). The nonlocal definition, i.e., formulation of the boundary conditions for Φ_1, does not simply require the indication that the solution is determined by the constant A and the parameters of the equation, but the complete formulation of the problem that is discarded in the approach proposed in the review. Analysis of the complete formulation of the Cauchy problem for (7) shows /1/ that for given $\varkappa, \varkappa_1$ the nontrivial solution Φ_1 does not at all exist for any previously assigned β. To determine the values of β needed, a nonlinear eigenvalue problem must be posed and solved, as is done in /1/. For the formal approach proposed in the review, this strictly determined number $\beta = \alpha$ remains only a guess if provided, of course, the answer is not known in advance.

To give the form of the solution, i.e., the exponent β, and to seek the specific equation of the problem (i.e., the ratio $\varkappa_1/\varkappa$) which this solution satisfies and for which this exponent is an eigenvalue is another problem. Despite what is said in the review article (p.262), this problem is not examined in the book /1/.

Let us emphasize that the separation of self-similar solutions into solutions of the first and second kinds is meaningful **only for a fixed formulation of the problem.** It is evident that by changing the problem, particularly the condition on the surface of discontinuity of the coefficient $\varkappa$ we also change the solution. I note this because it is explained in detail on pp.263 and 265 of the review that if one condition I took is replaced by another, then another solution is obtained. So what else is new?

As is seen, essentially different types of self-similar solutions of the Cauchy problem (6) and (7) are obtained for $\varkappa = \varkappa_1$ and $\varkappa \neq \varkappa_1$. In the first case the solution refers to self-similar solutions of the first kind, and in the second case to self-similar solutions of the second kind.

Special significance is given in the review (pp.264—265) to the trivial solutions for which the temperature, velocity, etc. are everywhere zero.

As applied to the heat conduction-filtration problem examined above, V. V. Markov writes (p.265):

"...it is possible to be occupied with a discussion of the purely mathematical questions of seeking solutions their and properties for problems formulated with mathematical formality.

From the mathematical viewpoint, in the linear (? — G. B.) formulation used by the author, his assertion that the self-similar problem of the removal of a finite mass of fluid at a point in filtration has no solution, is false. A trivial solution with the absence of perturbations is obtained in the formulation. The situation here is the same as in the problem considered above of taking account of radiation, where a solution also exists, but is trivial. **Trivial solutions are real and unique solutions, which are obtained as a result of the problem formulation used** (bold words are mine, G.B.), and it is not possible to say that they do not exist".

This assertion is incorrect. Indeed, let us turn to the mathematical formulation of the problem in the book /1/. On p.54 we read:

"Thus, a solution of equation (3.1) (equation (7) here, G. B.) is sought that satisfies the initial condition and the condition at infinity

$$u(x, 0) = 0 \quad (x \neq 0), \quad \int_{-\infty}^{\infty} u(x, 0)\, dx = Q; \quad u(\infty, t) = 0"$$

It is easy to see that the trivial solution

$$u(x, t) \equiv 0, \; -\infty \leqslant x \leqslant \infty, \; t \geqslant 0$$

does not satisfy the second of these conditions since for the trivial solution ... equals zero and not the positive quantity Q. The situation is analogous in the "problem of taking radiation into account," where the trivial solution

$$p \equiv 0, \; v \equiv 0, \; \rho \equiv \rho_0 \; (r \geqslant 0, \; t \geqslant 0)$$

does not satisfy the initial condition formulated "with mathematical formality" for extracting a finite amount of energy E at the center of an explosion at $t = 0$:

$$4\pi \int_0^{r_*} \rho \left[\frac{v^2}{2} + \frac{1}{\gamma - 1}\frac{p}{\rho}\right] r^2 dr = E \quad (t = 0)$$

(equation (2.19) on p.44 in the book /1/) since the integral on the left is zero and not $E > 0$ for this solution. Thus in both cases trivial solutions are not "actual solutions" of the formulated problems. Perhaps (see p.264 of the review) the function $u(x, t)$ which equals $Q\delta(x)$ for $t = 0$ and is identically zero for $t > 0$ is understood to be the trivial solution in the review? However, such a function is not a solution of (7) even in the generalized sense.

2. **Questions of priority.** It is asserted in the review (p.261) that questions of priority are resolved according to my desire on pp.70 and 82 of the book /1/, but "despite the facts". This assertion is refined on p.261:

"... It is impossible to ascribe credit to the authors of the important and interesting papers of Bechert and Guderley for the creation of a general theory of self-similar phenomena in gasdynamics nor in different applications of mathematics and physics.

...Thus in gas dynamics Barenblatt uses and discusses just those systems of ordinary equations and just those of their solutions that had already been given and studied by Sedov ..."

I do not ascribe credit to Bechert and Guderley for creating the general theory. Considering the self-similar solution of the problem of a strong explosion with losses or the influx of energy at the front, I wrote (p.70 of the book /1/): "The class of self-similar solutions of the gas dynamic equations to which the limiting solution (4.13) of this problem belongs was indicated by K. Bechert /70/ (*) and considered later by Sedov /59/ (**) and other authors".

As concerns Bechert, I was guided particulary by the footnote of Sedov (p.176 of the eighth edition of his book /4/). I cite it word for word:

"This class of solutions was mentioned in research of Sedov(***). Analogous solutions were examined in the paper of Bechert (****) without using dimensional analysis considerations or group theory and without relation to the formulation of the problems examined below(Bechert considers only polytropic motions).

Since the solutions are "analogous" and the Bechert paper is moreover dated 1941 and the Sedov paper 1945, its seemed natural to me to associate this class primarily with the name of Bechert.

Moreover, there are two substantial inaccuracies in the footnote cited. Bechert actually considered his solution from the viewpoint of group theory considerations that are identical to dimensional analysis (see p.360–361 of his paper /2/). Furthermore, Bechert also considered adiabatic motions with different entropies in the particles (see p.260 and subsequent pages), and not only barotropic polytropic processes. It is true, he examined these motions in detail only for the plane-waves case. However, this analysis had also been performed earlier in the Guderley paper /5/ for spherical and cylindrical waves.

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*) Reference /2/ in this paper.

**) Reference /3/ in this paper.

***) Reference /3/ follows.

****) Reference /2/ follows.

The 1942 paper of Guderley /5/, which is not cited in any paper of Sedov and in no edition of his book /4/, is discussed in the review. The content of this paper is inadequately elucidated in the review. In fact, in precisely this paper was the global self-similar problem of a spherical (and also cylindrical) strong shock converging to a center first formulated and solved. The Guderley paper /5/ is based completely on dimensional analysis and dimensionless parameters. Conditions on a strong shock (p.303) are first mentioned there. Precisely these conditions permitted inclusion of shock waves of variable intensity in the general system of self-similar unsteady gas motions. In 1946 Sedov essentially used these conditions in solving the problem of a strong explosion /6/. A family of self-similar spherically-symmetric (as well as cylindrically-symmetric) adiabatic motions with different entropies in the particles was indicated in the paper /5/ of Guderley, with suitable references to the above-mentioned paper of Bechert /2/, (p.303, equations (5a), (5b),(5c), which permitted Guderley to formulate and solve the above-mentioned global problem with shock waves, which is of practical importance. (In precisely this family is the solution I need, exactly as is the solution of the strong explosion problem). Furthermore, splitting of the system of ordinary equations for the self-similar solutions into one first order equation and two quadratures (p.304) was performed first in the Guderley paper /5/, and fields of integral curves of this equation, the "portraits" (pp. 305, 306 and 308, 309) were investigated; the solution of the above-mentioned global problem would be impossible without such an investigation. The absence of this Guderley paper /5/, known widely in the world literature (it is sufficient to mention the Russian translations of the texts by W. Hayes and R. Probstein /7/ and G. Whitham /8/) in all the papers and all the editions of the Sedov book /4/ is quite strange. In contrast, the Guderley paper /5/ is cited repeatedly in my book /1/. In particular, using the ordinary equations mentioned by the reviewer etc., I cited both authors (p.71): he who obtained first the result I needed, although more particular, Guderley, and he who subsequently proposed convenient, "notation, equations, qualitative schemes for their investigation" and exposition, Sedov. After reading the first version of the review, I arrived at the conclusion that the designation of this class of motions should actually be made more specific and called the Bechert— Guderley class, and not simply the Bechert class as on p.82 in /1/. I was able to do this in the English edition of my book /9/.

Furthermore, about the solution of the "short shock" problem supposedly "already long ascribed persistently to Zel'dovich". This solution was elucidated in 1963 in the monograph of Zel'dovich and Raiser /10/after research performed independently by Ia. B. Zel'dovich and his colleagues (1956), and is now commonly known. On p. 596 of this monograph the reader will find all the necessary references to the research of Weizsäcker and his colleagues that are mentioned on p.263 of the review. These papers are understandably also cited in my book /1/.

Prior to the publication of the book /1/, I did not know of the paper /11/ by Ia. G. Sapunkov, apparently because of the confusing title, "Convergent wave...", while the divergent waves of interest to me were considered there. The results of Sapunkov actually intercept those considered in my book for one value of the effective adiabatic index on the front $\gamma_1 = 2\gamma + 1$. I refer to the Sapunkov paper /11/ together with the already existing reference to the similar results obtained by A. Oppenheim and his colleagues in the English edition of my book /9/.

3. Individual comments. Let us turn to an examination of individual comments contained in the review.

It is mentioned on p.262 of the review that Fig.4.3 in the book /1/ does not fully describe the case $\gamma_1 = 2\gamma + 1$. In fact, in the caption to this figure, it is mentioned that it refers to the case $\gamma_1 > 2\gamma + 1$.

The figure presented in the review and referred to the case $\gamma_1 = 2\gamma + 1$ contains an inaccuracy. The lower segment of the heavy line by which the reviewer wished to enhance the "portrait" shrinks to a point for $\gamma_1 = 2\gamma + 1$. The fact is that the point of intersection of the curves, a singularity of node type lying to the upper left of the down-turned parabola in the reviewer's figure, lies on this parabola for $\gamma_1 = 2\gamma + 1$. (This is explained, in passing, on p.74 of the book /1/). The ambiguity of the solution for $\gamma_1 = 2\gamma + 1$ is specially noted on p.75 in /1/.

We read on p.265 and 266 of the review:

"In elucidating the hypotheses of local or complete isotropy of turbulent motions, it is impossible to bypass existing experimental data that directly contradict such hypotheses. In particular, for instance, it is found (a footnote refers to the papers of Compte— Bellot, Corrsin, Batchelor, and Stewart-G. B.) that behind grids the mean values of the longitudinal and transverse pulsations are different".

These data actually do not contradict the hypothesis of local isotropy, which refers to all developed turbulent flow. Local isotropy does not mean isotropy of the total velocity pulsations. Local isotropy means only isotropy of the field of relative velocities; this is also explained in the book /1/, (p.166).

As regards the "hypothesis" of complete isotropy, no one proposes it for any broad classes of flows. The beautiful experimental papers cited in the review show that not all flows are isotropic behind a grid. This is a well-known fact, noted on p.162 in the book /1/.

The reader will find the explicitly formulated assertion on p.150 of the book /1/: the assumption of the existence of a solution of the problem of ideal fluid flow past an infinite wedge is incorrect. Markov writes on p.266 that I do not remark and do not note the important circumstance of the nonexistence of this solution.

The necessity to take account of the resistance of viscous friction is noted on p.266 of the review, and it is mentioned that I do not do this on p.34 of the book /1/. If the reader will look at not only p.34 but also p.35 of the book /1/, he will find there the necessary words about viscous friction.

There are many such examples in the review, but I will not dwell on the remaining remarks: they are immaterial, and the reader will easily discern what the situation is by comparing what is written in the review with what is actually written in the book.

I will not dwell on the style of the review. However, I consider it necessary to note that it is deplorable to see unbridled expressions in the literature about Zel'dovich, an outstanding scientist, universallyknown for his research, a member of many academies, and thrice a Hero of Socialist Labor.

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